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ASYMPTOTIC EXPANSIONS OF THE LIAPUNOV INDEX FOR LINEAR STOCHASTIC SYSTEMS WITH SMALL NOISE*

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An expansion of the Liapunov index in powers of a small parameter is obtained for second-order with small noise for linear stochastic systems.

1. We shall investigate second-order linear stochastic systems of the form

$$dX^\varepsilon(t)/dt = BX^\varepsilon(t) + \varepsilon\sum_{r=1}^{k}\sigma_r X^\varepsilon(t)\,\xi_r^{\cdot}(t) \tag{1.1}$$

where $X^\varepsilon(t)$ is a two-dimensional column-vector, B and $\sigma_r = \|\sigma_r^{ij}\|$ are constant for order 2×2 -matrices, $\xi_r^{\cdot}(t)$ $(r = 1, 2,\ldots, k)$ are independent white noises, $\varepsilon > 0$ is a small parameter. System (1.1) will be understood in Ito's sense. Let $X^\varepsilon(t)$, $t\geqslant 0$ be a nontrivial solution of this system. In the non-degenerated (ergodic) case (/1/, Chapter VI) there exists a unique Liapunov index $\varkappa(\varepsilon)$ of system (1.1)

$$\varkappa(\varepsilon) = \lim_{t\to\infty}\frac{1}{t}M\ln|X^\varepsilon(t)|$$

It is well known /1/ that for asymptotic stability with probability one of the trivial solution of system (1.1) it is necessary and sufficient that the Liapunov index $\varkappa(\varepsilon)$ be negative. A method has been proposed in /1/ which in the case of second-order systems enables us to obtain an explicit formula for the Liapunov index. This method is connected with the construction of the density of an invariant measure on a circle for the process $X^\varepsilon(t)/|X^\varepsilon(t)|$, which satisfies a second-order linear differential equation solvable in quadratures. The Liapunov index $\varkappa(\varepsilon)$ is expressed as an integral of a known function with respect to this measure. The density $\mu^\varepsilon(\varphi)$ of the invariant measure satisfies the equation /1/

$$\frac{\varepsilon^2}{2}\frac{d^2}{d\varphi^2}[\Psi^2(\varphi)\mu^\varepsilon(\varphi)] - \frac{d}{d\varphi}[(H(\varphi)+\varepsilon^2F(\varphi))\mu^\varepsilon(\varphi)] = 0, \tag{1.2}$$

$$\varphi\in[0, 2\pi]$$

$$\Psi^2(\varphi) = \sum_{r=1}^{k}(\sigma_r\lambda(\varphi), \Lambda(\varphi))^2,\quad H(\varphi) = -(B\lambda(\varphi), \Lambda(\varphi))$$

$$F(\varphi) = \sum_{r=1}^{k}(\sigma_r\lambda(\varphi), \lambda(\varphi))(\sigma_r\lambda(\varphi), \Lambda(\varphi)),\ \lambda(\varphi) = \left\|\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\|,\quad \Lambda(\varphi) = \left\|\begin{matrix}\sin\varphi\\ -\cos\varphi\end{matrix}\right\|$$

the periodicity and norming conditions

$$\mu^\varepsilon(0) = \mu^\varepsilon(2\pi),\quad \int_0^{2\pi}\mu^\varepsilon(\varphi)\,d\varphi = 1 \tag{1.3}$$

Henceforth we assume that $\Psi^2(\varphi)\neq 0$ for all $\varphi\in[0, 2\pi]$. This ensures the ergodicity of process $X^\varepsilon(t)/|X^\varepsilon(t)|$. The Liapunov index $\varkappa(\varepsilon)$ is computed by the formula

$$\varkappa(\varepsilon) = \int_0^{2\pi}[(B\lambda(\varphi), \lambda(\varphi)) + \varepsilon^2L(\varphi)]\,\mu^\varepsilon(\varphi)\,d\varphi \tag{1.4}$$

$$L(\varphi) = \frac{1}{2}\sum_{r=1}^{k}(\sigma_r\lambda(\varphi), \sigma_r\lambda(\varphi)) - \sum_{r=1}^{k}(\sigma_r\lambda(\varphi), \lambda(\varphi))^2$$

The solution of Eq.(1.2) with the periodicity condition (the first relation in (1.3)) can be

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written as (/1/, Chapter VI)

$$\mu^\varepsilon(\varphi) = C(\varepsilon)\frac{\exp[-f(\varphi)/\varepsilon^2]}{\Psi^2(\varphi)\,\Omega(\varphi)}\frac{M(\varphi)}{M(0)} \tag{1.5}$$

$$M(\varphi) = \int_0^\pi \Omega(\varphi+\vartheta)\exp\frac{f(\varphi+\vartheta)}{\varepsilon^2}\,d\vartheta$$

$$\Omega(\varphi) = \exp\left[-2\int_0^\varphi \frac{F(\vartheta)}{\Psi^2(\vartheta)}\,d\vartheta\right], \quad f(\varphi) = -2\int_0^\varphi \frac{H(\vartheta)}{\Psi^2(\vartheta)}\,d\vartheta$$

The constant $C(\varepsilon)$ is determined from the norming condition (the second relation in (1.3)). We note the following properties of the above-mentioned functions, which can be verified directly:

$$\Psi^2(\varphi+\pi) = \Psi^2(\varphi),\ H(\varphi+\pi) = H(\varphi),\ F(\varphi+\pi) = F(\varphi),\ L(\varphi+\pi) = L(\varphi) \tag{1.6}$$

$$\Omega(\varphi+\pi) = \Omega(\varphi)\,\Omega(\pi),\quad f(\varphi+\pi) = f(\varphi)+f(\pi),\quad \mu^\varepsilon(\varphi+\pi) = \mu^\varepsilon(\varphi)$$

The question of the expansion of the Liapunov index in powers of a small parameter was touched upon in /2/ and is closely connected with the material in paper /3/ (also see the references /4,5/ mentioned in /3/). When the eigenvalues of matrix B are real and distinct, a detailed study of formula (1.5) shows that $\mu^\varepsilon(\varphi)$ is a delta-function and cannot be expanded in powers of ε. The explicit formula for $\varkappa(\varepsilon)$ contains a number of complex Laplace-type integrals. The case of complex eigenvalues of matrix B is simpler.

The main results of the present paper are simple explicit formulas for the first approximation of the asymptotic expansion of the Liapunov index $\varkappa(\varepsilon)$.

2. Case of real unequal eigenvalues of matrix B. Without loss of generality we can take it that $B = \mathrm{diag}(a, b)$, $a > b$. Using formula (1.5) for the density $\mu^\varepsilon(\varphi)$, we find $C(\varepsilon)$ from the norming condition in (1.3). Further, having substituted $\mu^\varepsilon(\varphi)$ into (1.4), we obtain

$$\varkappa(\varepsilon) = a - (a-b)\,I_1(\varepsilon)/I_0(\varepsilon) + \varepsilon^2 I_2(\varepsilon)/I_0(\varepsilon) \tag{2.1}$$

$$I_m(\varepsilon) = \int_0^\pi\int_0^\pi i_m(\varphi)\frac{\Omega(\varphi+\vartheta)}{\Psi^2(\varphi)\,\Omega(\varphi)}\exp\frac{g(\varphi,\vartheta)}{\varepsilon^2}\,d\varphi\,d\vartheta,\quad m=0,1,2$$

$$i_0(\varphi) = 1,\ i_1(\varphi) = \sin^2(\varphi),\ i_2(\varphi) = L(\varphi)$$

$$g(\varphi,\vartheta) \equiv f(\varphi+\vartheta) - f(\varphi) = (a-b)\int_\varphi^{\varphi+\vartheta}\frac{\sin 2t}{\Psi^2(t)}\,dt$$

We prove several auxiliary statements.

1°. As $\varepsilon \to 0$ the following asymptotic relations are valid:

$$I_m(\varepsilon) \sim J_m(\varepsilon) = P_m(\varepsilon)\,Q(\varepsilon),\quad P_m(\varepsilon) = \int_{-\pi/4}^{\pi/4}\frac{i_m(\varphi)\exp[-f(\varphi)/\varepsilon^2]}{\Psi^2(\varphi)\,\Omega(\varphi)}\,d\varphi \tag{2.2}$$

$$Q(\varepsilon) = \int_{\pi/4}^{3\pi/4}\Omega(t)\exp\frac{f(t)}{\varepsilon^2}\,dt,\quad m=0,1,2$$

Proof. We represent each of the integrals I_m as the sum $J_m + K_m$, where J_m are integrals of the corresponding integrands from (2.1), taken over the union of rhombuses $P_1 \cup P_2$ (Fig.1), while K_m are integrals of the same functions over the set $[0,\pi; 0,\pi] \setminus P_1 \cup P_2$, which we denote $\overline{P_1 \cup P_2}$. The integrals over each of the rhombuses P_1 and P_2 reduce, by the change of variables $t = \varphi + \vartheta$ for P_1 and $t = \varphi + \vartheta - \pi$ for P_2, to a product of two one-dimensional integrals. Thanks to the properties (1.6) of functions $\Psi^2(\varphi)$, $L(\varphi)$, $\Omega(\varphi)$ and $f(\varphi)$, after manipulations we obtain the representations indicated in (2.2) for J_m. Further, it can be established that the function $g(\varphi,\vartheta)$ in the square $[0,\pi; 0,\pi]$ takes a maximum value at two points: $(0, \pi/2) \in P_1$ and $(\pi, \pi/2) \in P_2$. According to Laplace's method /6/, these points make the principal contribution to the asymptotic behavior of integrals I_m for small values of ε. Hence follows the validity of the asymptotic representations (2.2).

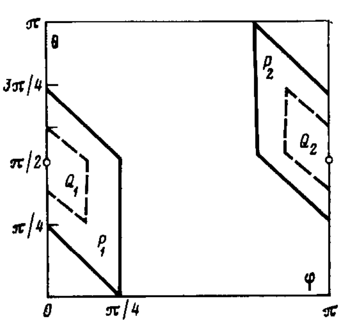

Fig.1

2°. As $\varepsilon \to 0$ the following asymptotic representation is valid for the Liapunov index:

$$\varkappa(\varepsilon) \sim a - (a-b) J_1(\varepsilon) / J_0(\varepsilon) + \varepsilon^2 J_2(\varepsilon) / J_0(\varepsilon) \tag{2.3}$$

Proof. Using formula (2.1), we write $\varkappa(\varepsilon)$ as:

$$\varkappa(\varepsilon) = a - (a-b) J_1/J_0 + \varepsilon^2 J_2/J_0 + \rho_0(\varepsilon)$$
$$\rho_0(\varepsilon) = (a-b)(J_1/J_0 - I_1/I_0) + \varepsilon^2 (I_0/I_2 - J_2/J_0) \tag{2.4}$$

We have

$$|J_1/J_0 - I_1/I_0| = |J_1K_0 - J_0K_1|/(J_0I_0) \leqslant 2K_0/I_0, \quad \text{since} \quad J_1 \leqslant J_0 \text{ and } K_1 \leqslant K_0$$

Analogously

$$|I_2/I_0 - J_2/J_0| \leqslant 2LK_0/I_0, \quad L = \max_{\varphi \in [0,\pi]} |L(\varphi)|, \text{ since } |K_2| \leqslant LK_0 \text{ and } J_2 | \leqslant LJ_0.$$

Therefore, we obtain the following inequality

$$|\rho_0(\varepsilon)| \leqslant 2(a-b+\varepsilon^2 L) K_0/I_0, \quad K_0 = \iint\limits_{\overline{P_1 \cup P_2}} \frac{\Omega(\varphi+\vartheta)}{\Psi^2(\varphi)\,\Omega(\varphi)} \text{ e p } \frac{g(\varphi,\vartheta)}{\varepsilon^2}\, d\varphi\, d\vartheta \tag{2.5}$$

We denote

$$g_1 = \max_{(\varphi,\vartheta)\in \overline{P_1 \cup P_2}} g(\varphi,\vartheta), \quad g_2 = \min_{(\varphi,\vartheta)\in Q_1\cup Q_2} g(\varphi,\vartheta), \quad \delta = g_2 - g_1$$

where Q_1 and Q_2 are closed regions in the square $[0,\pi;\ 0,\pi]$, such that $(0,\pi/2) \in Q_1$, $(\pi,\pi/2) \in Q_2$ (Fig.1). The function $g(\varphi,\vartheta)$ takes a maximum value at points $(0,\pi/2)$ and $(\pi,\pi/2)$; therefore, regions Q_1 and Q_2 can be found such that the inequality $g_2 > g_1$ is satisfied. Then

$$\frac{K_0}{I_0} \leqslant \frac{D}{2}\exp\left(-\frac{\delta}{\varepsilon^2}\right), \quad D = \frac{3\pi^2}{2S}\left(\frac{\Psi\Omega}{\psi\omega}\right)^2$$

$$\Psi^2 = \max_{\varphi\in[0,\pi]} \Psi^2(\varphi), \quad \Omega = \max_{\varphi\in[0,2\pi]} \Omega(\varphi), \quad \psi^2 = \min_{\varphi\in[0,\pi]} \Psi^2(\varphi), \quad \omega = \min_{\varphi\in[0,2\pi]} \Omega(\varphi)$$

(S is the area of the union of regions $Q_1 \cup Q_2$, $\delta > 0$). From the inequality just obtained and from inequality (2.5) results the following estimate:

$$|\rho_0(\varepsilon)| \leqslant D(a-b+\varepsilon^2 L)\exp\left(-\frac{\delta}{\varepsilon^2}\right) \tag{2.6}$$

Hence follows the asymptotic representation (2.3). The latter representation enables us to find an asymptotic expansion for $\varkappa(\varepsilon)$ in powers of ε^2. This expansion can be obtained by applying Laplace's method to each of the integrals occurring in (2.3). However, to obtain the first approximations and estimates for the remainders it is more convenient to expand in powers of ε^2 not the individual integrals but their ratios. For this we need auxiliary statements.

3°. Let $h(\varphi)$ be a function differentiable on $[-\pi/4, \pi/4]$, $h(0) = 0$. Then

$$\frac{|R(\varepsilon)|}{P_0(\varepsilon)} \leqslant A\varepsilon^2, \quad R(\varepsilon) = \int_{-\pi/4}^{\pi/4} h(\varphi)\exp\left[-\frac{f(\varphi)}{\varepsilon^2}\right] d\varphi, \tag{2.7}$$

$$A = (8\pi v_0 + v_1)\Psi^2\Omega$$

$$v_0 = \max_{\varphi\in[-\pi/4,\pi/4]} |v(\varphi)|, \quad v_1 = \max_{\varphi\in[-\pi/4,\pi/4]} \left|\frac{dv(\varphi)}{d\varphi}\right|,$$

$$v(\varphi) = \frac{h(\varphi)\Psi^2(\varphi)}{(a-b)\sin 2\varphi}$$

The expression for the constant A is obtained by an integration by parts of the integral for $R(\varepsilon)$.

4°. The following formulas are valid:

$$\frac{P_1(\varepsilon)}{P_0(\varepsilon)} = \varepsilon^2\frac{\Psi^2(0)}{2(a-b)} + \varepsilon^4\rho_1(\varepsilon), \quad \frac{P_2(\varepsilon)}{P_0(\varepsilon)} = L(0) + \varepsilon^2\rho_2(\varepsilon) \tag{2.8}$$

where $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ are estimated by some constants, uniformly with respect to ε. The first of formulas (2.8) is obtained by an integration by parts of the integral $P_1(\varepsilon)$ with the aid of certain elementary manipulations and of applying 3°. The second formula in (2.8) is obtained with the aid of analogous manipulations and of statement 3°. As a result, for $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ we have the estimates

$$|\rho_1(\varepsilon)| \leqslant \frac{4\Psi^4\Omega}{(a-b)^2\omega} + A_1, \quad |\rho_2(\varepsilon)| \leqslant A_2$$

where A_1 and A_2 are constants defined in 3°.

Theorem 1. The expansion

$$\varkappa(\varepsilon) = a - \frac{\varepsilon^2}{2}\sum_{r=1}^{k}(\sigma_r^{11})^2 + \varepsilon^4\rho(\varepsilon) + \rho_0(\varepsilon) \tag{2.9}$$

where $\rho(\varepsilon)$ is bounded in modulus by some constant independent of ε and $\rho_0(\varepsilon)$ satisfies inequality (2.6), holds for the Liapunov index of system (1.1).

The proof follows from (2.3), (2.4), statement 2°, (2.8) in statement 4°, and the easily verifiable equalities

$$\Psi^2(0) = \sum_{r=1}^{k}(\sigma_r^{21})^2, \quad L(0) = \frac{1}{2}\sum_{r=1}^{k}[(\sigma_r^{21})^2 - (\sigma_r^{11})^2] \tag{2.10}$$

$$|\rho(\varepsilon)| = |(a-b)\rho_1(\varepsilon) + \rho_2(\varepsilon)| \leqslant \frac{4\Psi^4\Omega}{(a-b)\omega} + (a-b)A_1 + A_2$$

3. Case of complex eigenvalues of matrix B. Without loss of generality we can take it that

$$B = \begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}$$

where $a, b > 0$ are real numbers. Then $H(\varphi) = -b$. The solution of Eq.(1.2) can be represented as

$$\mu^\varepsilon(\varphi) = \sum_{m=0}^{n-1}\varepsilon^{2m}u_m(\varphi) + \varepsilon^{2n}r_n^\varepsilon(\varphi) \tag{3.1}$$

After substitution into (1.2) and using the periodicity and norming conditions

$$u_m(0) = u_m(2\pi), \quad r_n^\varepsilon(0) = r_n^\varepsilon(2\pi)$$
$$\int_0^{2\pi} u_m(\varphi)\,d\varphi = \begin{cases} 1, & m=0 \\ 0, & m \neq 0 \end{cases}, \quad \int_0^{2\pi} r_n^\varepsilon(\varphi)\,d\varphi = 0, \quad m = 0, 1, \ldots, n-1 \tag{3.2}$$

we find

$$u_0(\varphi) = \frac{1}{2\pi}, \tag{3.3}$$
$$u_m(\varphi) = \frac{1}{b}\Big[F(\varphi)u_{m-1}(\varphi) - \frac{1}{2}\frac{d}{d\varphi}(\Psi^2(\varphi)u_{m-1}(\varphi)) - \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)u_{m-1}(\vartheta)\,d\vartheta\Big], \quad m = 1, 2, \ldots, n-1$$

$$r_n^\varepsilon(\varphi) = \frac{\exp[-f(\varphi)/\varepsilon^2]}{\Psi^2(\varphi)\Omega(\varphi)}\Big\{C_1(\varepsilon) + \frac{C_2(\varepsilon)}{\varepsilon^2}\int_0^\varphi \Omega(\vartheta)\exp\frac{f(\vartheta)}{\varepsilon^2}\,d\vartheta + \frac{2b}{\varepsilon^2}\int_0^\varphi u_n(\vartheta)\Omega(\vartheta)\exp\frac{f(\vartheta)}{\varepsilon^2}\,d\vartheta\Big\} \tag{3.4}$$

The constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are determined from conditions (3.2) for the functions $r_n^\varepsilon(\varphi)$. Using the explicit representation (3.4) for $r_n^\varepsilon(\varphi)$, we can obtain the estimate

$$|r_n^\varepsilon(\varphi)| \leqslant 4U_n\frac{\Psi^2\Omega}{\psi^2\omega}, \quad U_n = \max_{\varphi\in[0,\pi]}|u_n(\varphi)| \tag{3.5}$$

Theorem 2. The expansion

$$\varkappa(\varepsilon) = a + \sum_{m=1}^{n}\varepsilon^{2m}\int_0^{2\pi} L(\varphi)u_{m-1}(\varphi)\,d\varphi + \varepsilon^{2n+2}R_n(\varepsilon) \tag{3.6}$$

$$|R_n(\varepsilon)| = \Big|\int_0^{2\pi} L(\varphi)r_n^\varepsilon(\varphi)\,d\varphi\Big| \leqslant 8\pi U_n L\frac{\Psi^2\Omega}{\psi^2\omega}$$

holds for the Liapunov index of system (1.1). In particular, when $n = 1$

$$\varkappa(\varepsilon) = a + \frac{\varepsilon^2}{8} \sum_{r=1}^{k} [(\sigma_r^{12} - \sigma_r^{21})^2 - (\sigma_r^{11} + \sigma_r^{22})^2] + \varepsilon^4 R_1(\varepsilon) \tag{3.7}$$

Proof. Formula (3.6) follows from (3.1) and (1.4) (in this case $(B\lambda(\varphi), \lambda(\varphi)) = a$). The estimate for the remainder $R_n(\varepsilon)$ follows from inequality (3.5). Formula (3.7) is obtained from (3.6) by direct computations.

The following statement is valid:

5°. Let $\sigma_r^{22} = \sigma_r^{11}$ and $\sigma_r^{21} = -\sigma_r^{12}$, $r = 1, 2, \ldots, k$ (which is equivalent to the pairwise commutativity of the matrices $B, \sigma_1, \ldots, \sigma_k$). Then the exact formula

$$\varkappa(\varepsilon) = a + \frac{\varepsilon^2}{2} \sum_{r=1}^{k} [(\sigma_r^{12})^2 - (\sigma_r^{11})^2] \tag{3.8}$$

holds for the Liapunov index of system (1.1).

Proof. Under the conditions given, direct computations yield

$$\Psi^2(\varphi) = \sum_{r=1}^{k} (\sigma_r^{12})^2, \quad F(\varphi) = \sum_{r=1}^{k} \sigma_r^{11}\sigma_r^{12}$$

Therefore, from (3.3) with $m = 1$ we obtain

$$u_1(\varphi) = \frac{1}{2\pi b}\left[F(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)\, d\vartheta - \frac{1}{2}\frac{d}{d\varphi}(\Psi^2(\varphi))\right] = 0$$

Consequently, $U_1 = 0$ and $R_1(\varepsilon) = 0$ in formula (3.7); here (3.7) turns into (3.8).

4. Case of real equal eigenvalues of matrix B.

When $B = \mathrm{diag}\,(a, a)$, $H(\varphi) = 0$ and the density $\mu(\varphi)$ is independent of ε (see Eq. (1.2)). Therefore, the formula for the Liapunov index of system (1.1) becomes

$$\varkappa(\varepsilon) = a + \varepsilon^2 \int_0^{2\pi} L(\varphi)\mu(\varphi)\, d\varphi \tag{4.1}$$

where $\mu(\varphi)$ satisfies the equation

$$\frac{1}{2}\frac{d^2}{d\varphi^2}[\Psi^2(\varphi)\mu(\varphi)] - \frac{d}{d\varphi}[F(\varphi)\mu(\varphi)] = 0$$

and the conditions (1.3). For the complex case when

$$B = \begin{Vmatrix} a & 1 \\ 0 & a \end{Vmatrix} \tag{4.2}$$

without dwelling on the very cumbersome calculations we present the asymptotic formula

$$\varkappa(\varepsilon) = a + \varepsilon^{2/3} \frac{\pi^{1/2}}{\Gamma(1/6)} \left[\frac{3}{4}\sum_{r=1}^{k} (\sigma_r^{21})^2\right]^{1/3} + O(\varepsilon) \tag{4.3}$$

5. The Liapunov index for Stratonovich-systems.

Let us extend the results to Stratonovich-systems of stochastic equations

$$\frac{dX^\varepsilon(t)}{dt} = BX^\varepsilon(t) + \varepsilon \sum_{r=1}^{k} \sigma_r X^\varepsilon(t)\, \dot{\xi}_r^*(t) \tag{5.1}$$

The formulas for the Liapunov index of system (5.1), which we denote $\varkappa^*(\varepsilon)$, are obtained from the arguments and formulas presented above if the functions $F(\varphi)$ and $L(\varphi)$ are replaced, respectively, by the functions

$$F(\varphi) - (B_0\lambda(\varphi), \Lambda(\varphi)),\ L(\varphi) + (B_0\lambda(\varphi), \lambda(\varphi)) \quad \left(B_0 = \frac{1}{2}\sum_{r=1}^{k} \sigma_r^2\right)$$

Let us write out the definitive formulas for $\varkappa^*(\varepsilon)$ (the matrix B below has the previous form).

In the case of real unequal and complex eigenvalues of matrix B we have, respectively,

$$\varkappa^*(\varepsilon) = a + \frac{\varepsilon^2}{2}\sum_{r=1}^{k}\sigma_r^{12}\sigma_r^{21} + \varepsilon^4\rho^*(\varepsilon) + \rho_0^*(\varepsilon) \qquad (5.2)$$

$$\varkappa^*(\varepsilon) = a + \frac{\varepsilon^2}{8}\sum_{r=1}^{k}[(\sigma_r^{12}+\sigma_r^{21})^2 + (\sigma_r^{11}-\sigma_r^{22})^2] + \varepsilon^4 R_1^*(\varepsilon) \qquad (5.3)$$

Here $\rho_0^*(\varepsilon)$ and $\rho^*(\varepsilon)$ are estimated analogously to $\rho_0(\varepsilon)$ and $\rho(\varepsilon)$ (see (2.6), (2.10) with due regard to the changes in the functions indicated; $R_1^*(\varepsilon)$ is estimated analogously to $R_1(\varepsilon)$ in formula (3.7) with due regard to what has been said. If in the latter case $\sigma_r^{22} = \sigma_r^{11}$, $\sigma_r^{21} = -\sigma_r^{12}$, $r = 1, 2, \ldots, k$, then there holds the exact formula

$$\varkappa^* = a \qquad (5.4)$$

In case $B = \mathrm{diag}(a, a)$

$$\varkappa^*(\varepsilon) = a + \varepsilon^2\int_0^{2\pi}[L(\varphi) + (B_0\lambda(\varphi), \lambda(\varphi))]\mu^*(\varphi)\,d\varphi \qquad (5.5)$$

where $\mu^*(\varphi)$ is the solution of the equation

$$\frac{1}{2}\frac{d^2}{d\varphi^2}[\Psi^2(\varphi)\mu^*(\varphi)] - \frac{d}{d\varphi}\{[F(\varphi) - (B_0\lambda(\varphi), \Lambda(\varphi))]\mu^*(\varphi)\} = 0$$

satisfying conditions (1.3). In the case of (4.2) formula (4.3) holds for $\varkappa^*(\varepsilon)$.

Example 1. A Stratonovich-system is asymptotically stable with probability one while the Itô-system is unstable. Let

$$B = \begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}, \quad a < 0, \quad k = 1, \quad \sigma_1 = \begin{Vmatrix} 0 & 2\sqrt{-a} \\ -2\sqrt{-a} & 0 \end{Vmatrix}, \quad \varepsilon = 1$$

Then (see (3.8) and (5.4)) $\varkappa = -a > 0, \varkappa^* = a < 0$.

Example 2. An unstable deterministic system becomes asymptotically stable when specific noises are imposed, understood both in the sense of Itô and in the sense of Stratonovich. Let

$$B = \mathrm{diag}(a, b), \quad a > b, \ k = 2, \quad \sigma_1 = \sigma\begin{Vmatrix} 1 & 1 \\ -1 & 1 \end{Vmatrix}, \quad \sigma_2 = \sigma\begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix}, \quad \sigma > 0$$

We have

$$\Psi^2(\varphi) = 2\sigma^2, \ F(\varphi) = 0, \ L(\varphi) = 0, \ \Omega(\varphi) = 1.$$

Therefore, here $\varkappa^*(\varepsilon) = \varkappa(\varepsilon)$ and (see (2.9))

$$\varkappa(\varepsilon) = a - \varepsilon^2\sigma^2 + \varepsilon^4\rho(\varepsilon) + \rho_0(\varepsilon)$$

As the regions Q_1 and Q_2 we take the rhombuses shown in the Fig.1. Then the sum of their areas is $S = \pi^2/16$ and

$$g_1 = \frac{a-b}{4\sigma^2}, \quad g_2 = \frac{a-b}{4\sigma^2}\sqrt{2}, \quad |\rho_0(\varepsilon)| \leqslant 24(a-b)\exp\left[-\frac{(a-b)(\sqrt{2}-1)}{4\sigma^2\varepsilon^2}\right]$$

i.e. $D = 24$, $\delta = (a-b)(\sqrt{2}-1)/4\sigma^2$. Taking into account the last inequality, we obtain

$$|\rho_0(\varepsilon)| \leqslant \frac{1536(3+2\sqrt{2})\sigma^4}{e^2(a-b)}\varepsilon^4$$

We estimate $\rho(\varepsilon)$ on the strength of (2.10)

$$|\rho(\varepsilon)| \leqslant \frac{4(4/\pi+6)\sigma^4}{a-b}$$

The following estimate is obtained for $\varkappa(\varepsilon)$

$$a - \varepsilon^2\sigma^2 - \varepsilon^4\sigma^4\frac{3328}{a-b} < \varkappa(\varepsilon) < a - \varepsilon^2\sigma^2 + \varepsilon^4\sigma^4\frac{3328}{a-b}$$

Hence it is clear that if $a > 0$, then for negative values of b sufficiently large in absolute value (say, $b < -2\cdot 10^4 a$) and for a specified σ we can find an interval of values of ε, in which $\varkappa(\varepsilon) < 0$. Thus, unstable systems exist which under imposed noises become asymptotically stable. In the example given, if $a = 0, b < 0$, then for $0 < \varepsilon < 0.01\sqrt{-b}/\sigma$ we obtain $\varkappa(\varepsilon) < 0$, i.e., asymptotic stability holds.

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